1. Vectors and Parametric Equation of a Line

In this lecture, we will discuss

- Vectors
  - Cartesian Coordinates and Polar Coordinates
  - Length of a vector; Addition of Vectors and Multiplication by Scalars
- Parametric Curves

Vectors

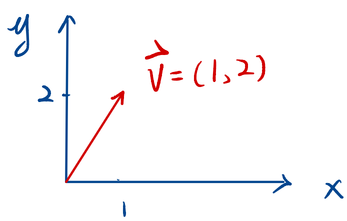
Below, we review the definitions and basic properties of vectors.

Cartesian Coordinates

- In $\mathbb{R}^2$,
  - A vector $\mathbf{v} = (v_1, v_2)$ in $\mathbb{R}^2$ can be written as

$$\mathbf{v} = (v_1, v_2) = v_1(1,0) + v_2(0,1) = v_1\mathbf{i} + v_2\mathbf{j},$$

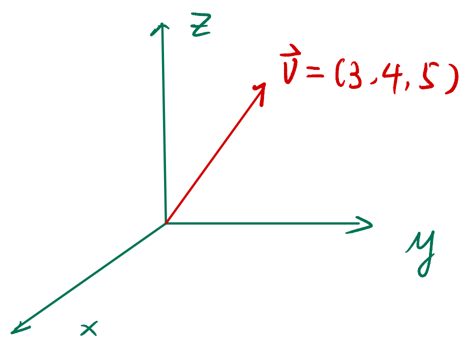
where $\mathbf{i} = (1,0), \mathbf{j} = (0,1)$ are the standard unit vectors in $\mathbb{R}^2$.



- In $\mathbb{R}^3$,
  - A vector $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ can be written as

$$\mathbf{v} = (v_1, v_2, v_3) = v_1(1,0,0) + v_2(0,1,0) + v_3(0,0,1) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k},$$

where $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, and $\mathbf{k} = (0,0,1)$ are the standard unit vectors in $\mathbb{R}^3$.



- Generally, $\mathbb{R}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{R}, i = 1, \ldots, n\}$.

- **Polar and Cartesian coordinates**

  We compare polar and Cartesian coordinates as follows:

  - Place the pole at the origin and the polar axis over the positive direction of the x-axis.
  - Note

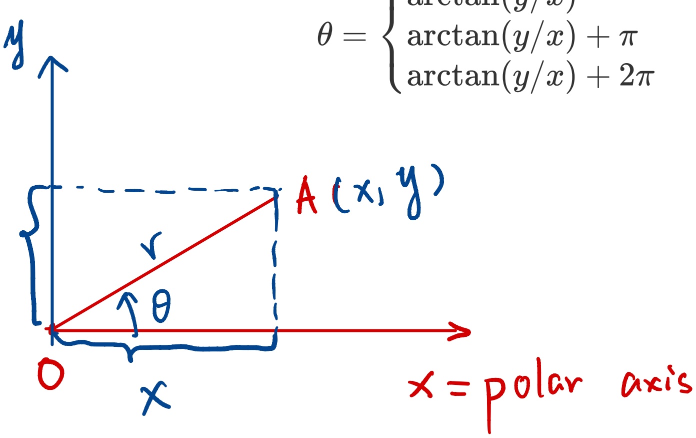
  $$x = r\cos\theta, \quad y = r\sin\theta$$

  - If x and y are known,

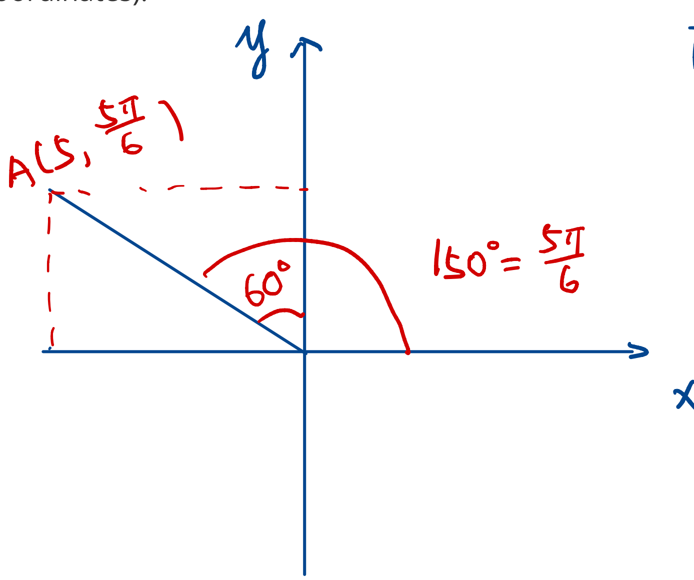
  $$r = \sqrt{x^2+y^2}, \quad \tan\theta = y/x, \quad 0 \le \theta < 2\pi$$

    give corresponding polar coordinates.
  - Notice that $\arctan(y/x) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, but the requirement for θ is $0 \le \theta < 2\pi$. We have

  $$\theta = \begin{cases} \arctan(y/x) & \text{if } x > 0 \text{ and } y \ge 0 \\ \arctan(y/x) + \pi & \text{if } x < 0 \\ \arctan(y/x) + 2\pi & \text{if } x > 0 \text{ and } y < 0 \end{cases}$$



Example 1. Find the vector of length 5 making an angle of 60° with the y-axis (present the vector in Cartesian Coordinates).



From the graph, we know

$$x = -5 \cdot \cos\frac{\pi}{6} = -\frac{5\sqrt{3}}{2}$$

$$y = 5 \cdot \sin\frac{\pi}{6} = \frac{5}{2}$$

Length of a vector

The length of a vector is equal to the length of any of its representatives.

If $\mathbf{v} = (v_1, v_2)$ is a vector in $\mathbb{R}^2$, then $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$.

If $\mathbf{v} = (v_1, v_2, v_3)$ is a vector in $\mathbb{R}^3$, then $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

- **Unit Vector**
  - A vector whose length is 1 is called a unit vector.
  - If $\mathbf{v}$ is a nonzero vector, then the vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ is the unit vector in the same direction as $\mathbf{v}$.
  - Constructing a unit vector $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ from a nonzero vector $\mathbf{v}$ is sometimes called *normalizing a vector*.

Example 2.

1. Find the length of the vector $\mathbf{v}$.
2. Find the vector parallel to $\mathbf{v}$ with length 2.

$$\mathbf{v} = \sin\theta\mathbf{i} + \cos\theta\mathbf{j} + \mathbf{k}$$

ANS: 1. By def, $\|\vec{v}\| = \sqrt{\sin^2\theta + \cos^2\theta + 1^2} = \sqrt{2}$

2. We first compute the unit vector in the direction of $\vec{v}$

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{\sin\theta\,\vec{i} + \cos\theta\,\vec{j} + \vec{k}}{\sqrt{2}}$$

Thus the vector in the same direction as $\vec{v}$ with length 2 is

$$\frac{2\vec{v}}{\|\vec{v}\|} = \sqrt{2}\sin\theta\,\vec{i} + \sqrt{2}\cos\theta\,\vec{j} + \sqrt{2}\,\vec{k}$$

Addition of Vectors and Multiplication by Scalars

Definitions (Addition and Scalar Multiplication)

(a) (Addition of Vectors) The sum $\mathbf{v}+\mathbf{w}$ and the difference $\mathbf{v}-\mathbf{w}$ of two vectors $\mathbf{v}=(v_1,v_2)$ and $\mathbf{w}=(w_1,w_2)$ in $\mathbb{R}^2$ are the vectors given by $\mathbf{v}+\mathbf{w}=(v_1+w_1,v_2+w_2)$ and $\mathbf{v}-\mathbf{w}=(v_1-w_1,v_2-w_2)$.

If $\mathbf{v}=(v_1,v_2,v_3)$ and $\mathbf{w}=(w_1,w_2,w_3)$ are in $\mathbb{R}^3$, then $\mathbf{v}+\mathbf{w}=(v_1+w_1,v_2+w_2,v_3+w_3)$ and $\mathbf{v}-\mathbf{w}=(v_1-w_1,v_2-w_2,v_3-w_3)$.

(b) (Scalar Multiplication) If $\mathbf{v}=(v_1,v_2)\in\mathbb{R}^2$ and $\alpha\in\mathbb{R}$, then $\alpha\mathbf{v}$ is the vector in $\mathbb{R}^2$ defined by $\alpha\mathbf{v}=(\alpha v_1,\alpha v_2)$. If $\mathbf{v}=(v_1,v_2,v_3)$, then $\alpha\mathbf{v}=(\alpha v_1,\alpha v_2,\alpha v_3)$ for any real number α.

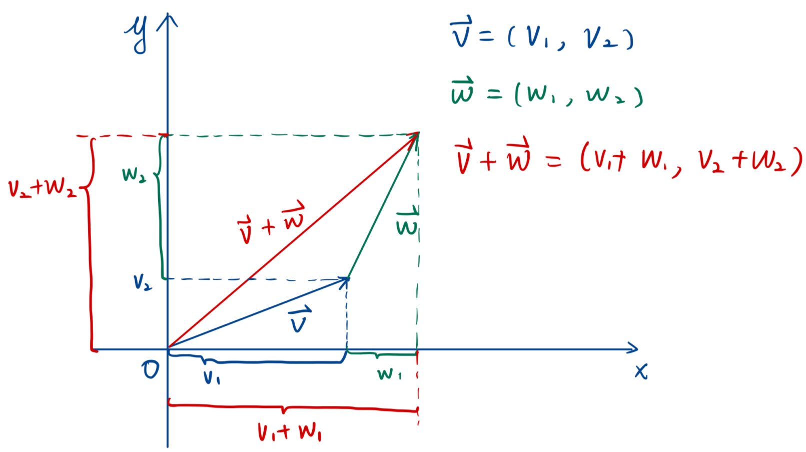
Remark (Parallel Vectors). We say the vectors $\mathbf{v}$ and $\mathbf{w}$ are parallel if there exist a nonzero number α such that $\mathbf{w}=\alpha\mathbf{v}$.

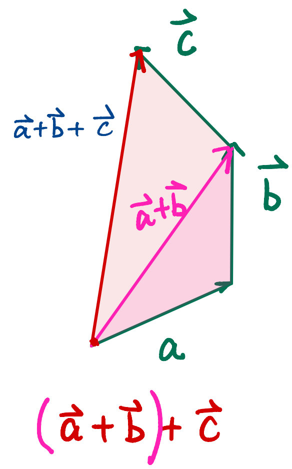
If $\alpha>0$, then $\alpha\mathbf{v}$ and $\mathbf{v}$ have the same direction.

If $\alpha<0$, then $\alpha\mathbf{v}$ and $\mathbf{v}$ have the opposite direction.

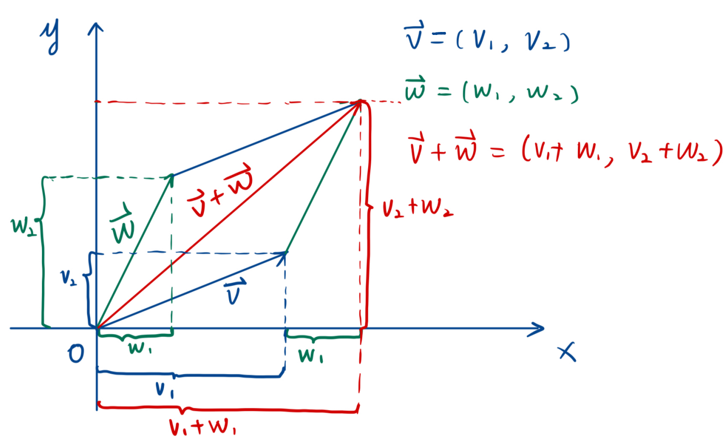
- **Triangle Law**

  Triangle Inequality: $\|\mathbf{v}+\mathbf{w}\|\le\|\mathbf{v}\|+\|\mathbf{w}\|$





- **Parallelogram Law**



THEOREM 1.1 Properties of Addition and Multiplication by Scalars

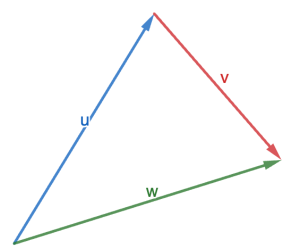
For all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^2$ (or, for all vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ in $\mathbb{R}^3$) and real numbers α and β, the following properties hold:

$$\mathbf{v}+\mathbf{w}=\mathbf{w}+\mathbf{v} \qquad \text{(commutativity)}$$
$$\mathbf{u}+(\mathbf{v}+\mathbf{w})=(\mathbf{u}+\mathbf{v})+\mathbf{w} \qquad \text{(associativity)}$$
$$\alpha(\mathbf{v}+\mathbf{w})=\alpha\mathbf{v}+\alpha\mathbf{w} \qquad \text{(distributivity)}$$
$$(\alpha+\beta)\mathbf{v}=\alpha\mathbf{v}+\beta\mathbf{v} \qquad \text{(distributivity)}$$
$$(\alpha\beta)\mathbf{v}=\alpha(\beta\mathbf{v}).$$

If $\mathbf{0}$ denotes the zero vector, then $\mathbf{v}+\mathbf{0}=\mathbf{v}$. Finally, $1\cdot\mathbf{v}=\mathbf{v}$.

Example 3.

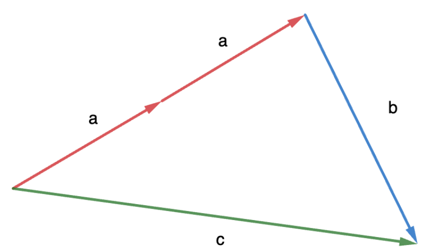
(1) Write the vector $\mathbf{u}$ in terms of the other vectors.



$\vec{u} = \vec{w} - \vec{v}$

(2) Write the vector $\mathbf{b}$ in terms of the other vectors.

$\vec{b} = -2\vec{a} + \vec{c}$

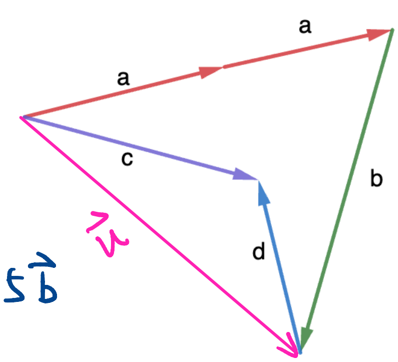


(3) Write the vector $\mathbf{a}$ in terms of the other vectors.

Notice that

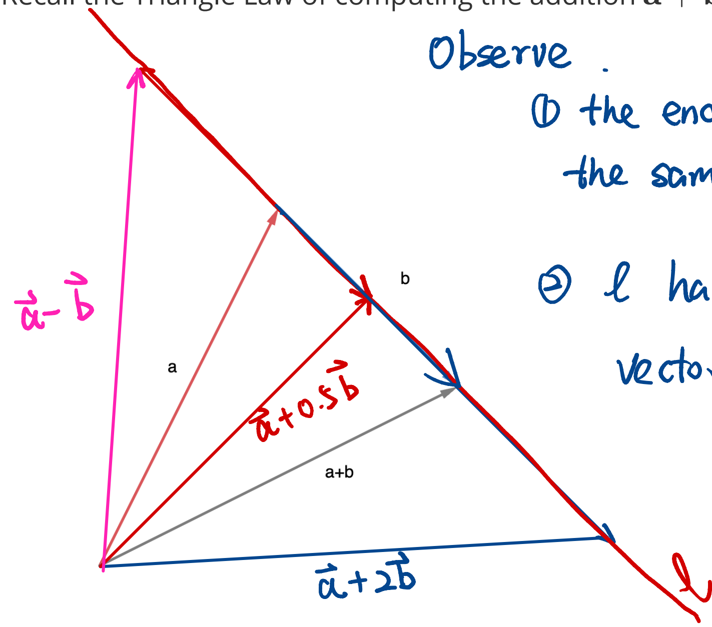
$2\vec{a} = \vec{c} - \vec{d} - \vec{b}$

$\vec{a} = 0.5\,\vec{c} - 0.5\,\vec{d} - 0.5\,\vec{b}$



Parametric Equation of a Line

Recall the Triangle Law of computing the addition $\mathbf{a} + \mathbf{b}$.

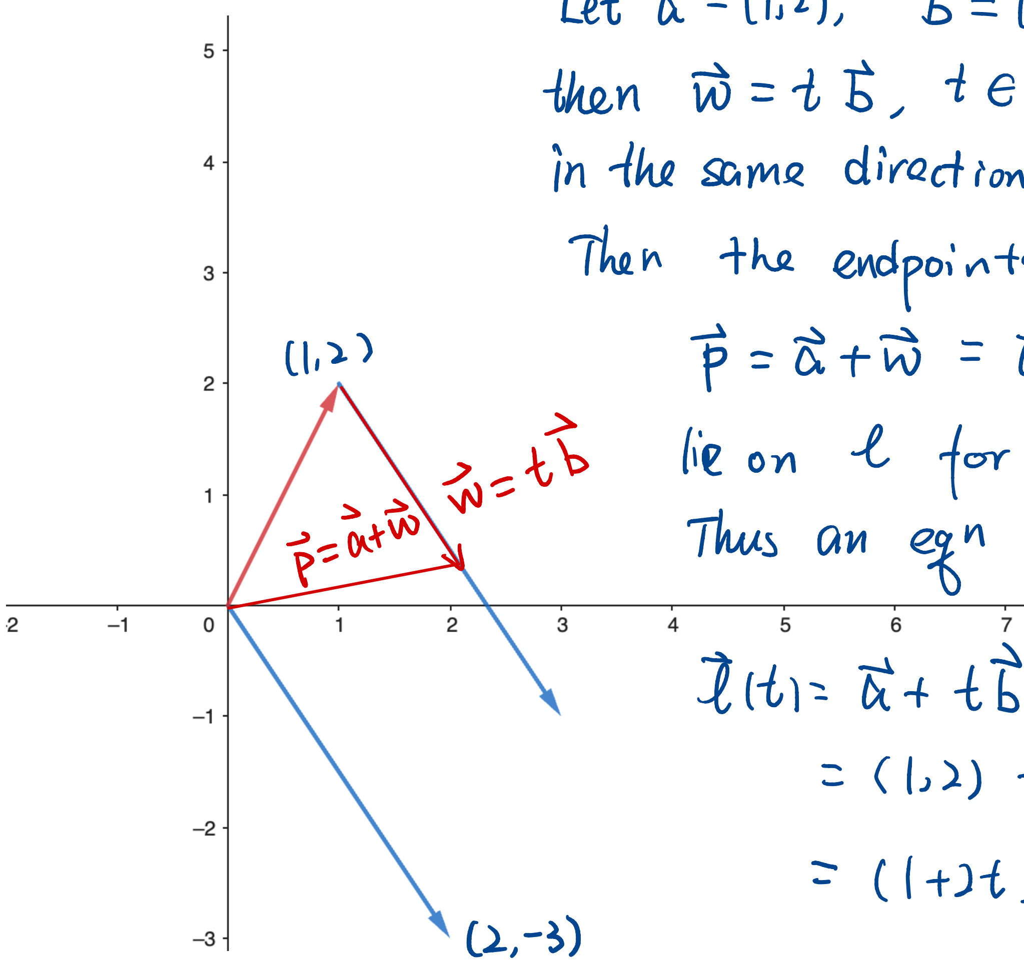


Observe.

① the endpoints of $\vec{a} + t\vec{b}$ lie on the same line ℓ. ($t \in \mathbb{R}$)

② ℓ has the same direction as vector $\vec{b}$.

Example 4. Find an equation of the line ℓ in $\mathbb{R}^2$ that passes through $(1, 2)$ and in the direction of the vector $(2, -3)$.



Let $\vec{a} = (1,2)$, $\vec{b} = (2,-3)$

then $\vec{w} = t\vec{b}$, $t \in \mathbb{R}$ is a vector in the same direction of $\vec{b}$.

Then the endpoints of

$$\vec{p} = \vec{a} + \vec{w} = \vec{a} + t\vec{b}$$

lie on ℓ for any $t \in \mathbb{R}$.

Thus an eqn for ℓ is

$$\vec{\ell}(t) = \vec{a} + t\vec{b}$$

$$= (1,2) + t(2,-3)$$

$$= (1+2t, 2-3t)$$

Summary. Parametric Equation of a Line

1. Pick a point $A\,(a_1, a_2)$ and a vector $\mathbf{v} = (v_1, v_2)$.
2. Let ℓ denote the line that contains A and whose direction is the same as $\mathbf{v}$, and let $\boldsymbol{P}(x, y)$ be a point on it.
3. By the Triangle Law, $\mathbf{p} = \mathbf{a} + \mathbf{w}$, where $\mathbf{p} = (x, y)$, $\mathbf{a} = (a_1, a_2)$, and $\mathbf{w}$ is the vector from A to P.
4. Since $\mathbf{w}$ is parallel to $\mathbf{v}$, $\mathbf{w} = t\mathbf{v}$ for some $t \in \mathbb{R}$
5. Thus $\mathbf{p} = \mathbf{a} + t\mathbf{v}, t \in \mathbb{R}$, which is the vector form of a parametric equation of the line ℓ.
6. This equation is usually written as

$$\mathbf{l}(t) = \mathbf{a} + t\mathbf{v}, \quad t \in \mathbb{R},$$

or,

$$\mathbf{l}(t) = (a_1 + tv_1, a_2 + tv_2), \quad t \in \mathbb{R}, \text{ or,}$$

Any of the above forms is called a parametric equation (or parametric equations) of a line.

In $\mathbb{R}^3$, the parametric equations of the line ℓ that contains a point $A\,(a_1, a_2, a_3)$ and with direction of a vector $\mathbf{v} = (v_1, v_2, v_3)$ are

$$\mathbf{l}(t) = \mathbf{a} + t\mathbf{v} = (a_1 + tv_1, a_2 + tv_2, a_3 + tv_3), \quad t \in \mathbb{R}.$$

Exercise 5.

1. Find an equation $\mathbf{r}(t)$ of the line in $\mathbb{R}^3$ that contains $(1, 2, 0)$ and $(0, -2, 4)$.
2. Rewrite $\mathbf{r}(t)$ as the corresponding parametric equations for the line:
   $x(t) = ?$
   $y(t) = ?$
   $z(t) = ?$

ANS: (1) Since points $A = (1,2,0)$ $B = (0,-2,4)$ are on the line ℓ, ℓ has the same direction as $\overrightarrow{AB} = (-1, -4, 4)$

Thus we can treat ℓ as a line that contains

$A = (1,2,0)$ and the direction

$\overrightarrow{AB} = (-1, -4, 4)$

Thus $\vec{r}(t) = (1, 2, 0) + t(-1, -4, 4)$

$= (1-t, 2-4t, 4t)$

(2) The corresponding

$x(t) = 1-t$

$y(t) = 2-4t$

$z(t) = 4t$.